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DISCUSSION

P. V. Lade[1] (*written discussion*)—The author has presented a comprehensive review of torsion shear testing of soils. Experimental observations discussed by the author suggest that the two horizontal, normal strains in a hollow cylinder exposed to torsional shear stresses may be equal in magnitude. It may, however, be inappropriate to set these strains equal a priori. This is demonstrated by the following theoretical derivations.

The stresses acting in the hollow cylinder specimen employed in torsion shear tests are shown in Fig. A. Equilibrium in the radial direction expressed in polar coordinates for an element of the specimen requires:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

in which r = radial distance to a point in the hollow cylinder, σ_r = radial normal stress, and σ_θ = tangential normal stress. The horizontal strain–displacement equations are:

$$\epsilon_r = -\frac{\partial u_r}{\partial r} \tag{2}$$

$$\epsilon_\theta = -\frac{u_r}{r} \tag{3}$$

in which u_r = radial displacement.

Elastic Behavior

For isotropic elastic behavior of soils, Hooke's law provides the following expressions for the horizontal normal stresses:

$$\sigma_r = (\lambda + 2G) \cdot \epsilon_r + \lambda \cdot \epsilon_\theta + \lambda \cdot \epsilon_z \tag{4}$$

$$\sigma_\theta = \lambda \cdot \epsilon_r + (\lambda + 2G) \cdot \epsilon_\theta + \lambda \cdot \epsilon_z \tag{5}$$

in which λ = Lame's constant and G = shear modulus. Since ϵ_z is not a function of the

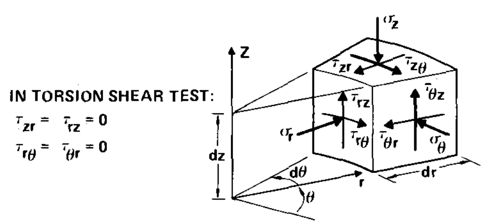

FIG. A—*Stresses acting in hollow cylinder specimen employed in torsion shear tests.*

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radius in a torsion shear tests, Eq 4 yields:

$$\frac{\partial \sigma_r}{\partial r} = (\lambda + 2G) \cdot \frac{\partial \epsilon_r}{\partial r} + \lambda \cdot \frac{\partial \epsilon_\theta}{\partial r} \tag{6}$$

Substitution of Eqs 2 to 6 into Eq 1 produces

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} = 0 \tag{7}$$

in which substitution of Eqs 2 and 3 yields:

$$-\frac{\partial \epsilon_r}{\partial r} - \frac{\epsilon_r - \epsilon_\theta}{r} = 0 \tag{8}$$

This equation is fulfilled by an elastic hollow cylinder exposed to inside and outside pressures as well as torsional shear stresses. For a thin-walled hollow cylinder the variation in radial strain over the wall thickness is negligible. Equation 8 shows that in this case of elastic behavior the two horizontal normal strains are equal.

Plastic Behavior

Since the relative magnitudes of the horizontal, normal strains are being considered, only the plastic potential function and its derivatives need be considered. For an isotropic material the plastic potential function may be expressed in terms of stress invariants:

$$g = g(I_1, I_2, I_3) \tag{9}$$

in which I_1, I_2, and I_3 are the invariants of the stress tensor:

$$I_1 = \sigma_r + \sigma_\theta + \sigma_z \tag{10}$$

$$I_2 = \tau_{r\theta} \cdot \tau_{\theta r} + \tau_{\theta z} \cdot \tau_{z\theta} + \tau_{zr} \cdot \tau_{rz} - (\sigma_r \cdot \sigma_\theta + \sigma_\theta \cdot \sigma_z + \sigma_z \cdot \sigma_r) \tag{11}$$

$$\begin{aligned} I_3 = {} & \sigma_r \cdot \sigma_\theta \cdot \sigma_z + \tau_{r\theta} \cdot \tau_{\theta z} \cdot \tau_{zr} + \tau_{\theta r} \cdot \tau_{z\theta} \cdot \tau_{rz} \\ & - (\sigma_r \cdot \tau_{\theta z} \cdot \tau_{z\theta} + \sigma_\theta \cdot \tau_{zr} \cdot \tau_{rz} + \sigma_z \cdot \tau_{r\theta} \cdot \tau_{\theta r} \end{aligned} \tag{12}$$

Other stress invariants which encompass similar effects as those in Eqs 10–12 may also be employed.

According to St. Venant's observations the plastic strain increments in the horizontal directions may be written as:

$$\dot{\epsilon}_r^p = \dot{\lambda} \cdot \frac{\partial g}{\partial \sigma_r} \tag{13}$$

$$\dot{\epsilon}_\theta^p = \dot{\lambda} \cdot \frac{\partial g}{\partial \sigma_\theta} \tag{14}$$

in which $\dot{\lambda}$ is a scalar and

$$\frac{\partial g}{\partial \sigma_r} = \frac{\partial g}{\partial I_1} \cdot \frac{\partial I_1}{\partial \sigma_r} + \frac{\partial g}{\partial I_2} \cdot \frac{\partial I_2}{\partial \sigma_r} + \frac{\partial g}{\partial I_3} \cdot \frac{\partial I_3}{\partial \sigma_r} \tag{15}$$

$$\frac{\partial g}{\partial \sigma_\theta} = \frac{\partial g}{\partial I_1} \cdot \frac{\partial I_1}{\partial \sigma_\theta} + \frac{\partial g}{\partial I_2} \cdot \frac{\partial I_2}{\partial \sigma_\theta} + \frac{\partial g}{\partial I_3} \cdot \frac{\partial I_3}{\partial \sigma_\theta} \tag{16}$$

The derivatives of the three stress invariants with regard to the normal stresses σ_r and σ_θ are:

$$\frac{\partial I_1}{\partial \sigma_r} = 1 \tag{17}$$

$$\frac{\partial I_1}{\partial \sigma_\theta} = 1 \tag{18}$$

$$\frac{\partial I_2}{\partial \sigma_r} = -(\sigma_\theta + \sigma_z) = \sigma_r - I_1 \tag{19}$$

$$\frac{\partial I_2}{\partial \sigma_\theta} = -(\sigma_r + \sigma_z) = \sigma_\theta - I_1 \tag{20}$$

$$\frac{\partial I_3}{\partial \sigma_r} = \sigma_\theta \cdot \sigma_z - \tau_{\theta z} \cdot \tau_{z\theta} \tag{21}$$

$$\frac{\partial I_3}{\partial \sigma_\theta} = \sigma_r \cdot \sigma_z - \tau_{zr} \cdot \tau_{rz} \tag{22}$$

The coefficients $\partial g/\partial I_1$, $\partial g/\partial I_2$, and $\partial g/\partial I_3$ are the same in Eqs 15 and 16. The coefficients $\partial I_1/\partial \sigma_r$ and $\partial I_1/\partial \sigma_\theta$ in these two equations are also the same (Eqs 17 and 18). For a test on a thin-walled hollow cylinder in which the inside and outside pressures are equal, equilibrium requires that $\sigma_r = \sigma_\theta$. For this case the coefficients $\partial I_2/\partial \sigma_r$ and $\partial I_2/\partial \sigma_\theta$ also have equal values (Eqs 19 and 20). Thus, if the plastic potential function g (Eq 9) is a function of I_1 and I_2 only, then the two strain increments $\dot{\epsilon}_r^p$ and $\dot{\epsilon}_\theta^p$ would be equal for the isotropic plastic material. Since the elastic and plastic strains are usually considered to be additive, an isotropic elasto-platic material for which $g = g\ (I_1,I_2)$ would always produce equal horizontal normal strains, $\epsilon_r = \epsilon_\theta$. For such a material it would not be necessary to measure the horizontal strains, since they could be calculated from the vertical and the volumetric strains, as in the triaxial test: $\epsilon_r = \epsilon_\theta = \frac{1}{2} \cdot (\epsilon_v - \epsilon_1)$.

In the general case the plastic potential function may depend on all three stress invariants. If torsional shear stress were not applied to the hollow cylinder, the coefficients $\partial I_3/\partial \sigma_r$ and $\partial I_3/\partial \sigma_\theta$ would also be the same, as long as the inside and outside pressures were equal, resulting in $\sigma_r = \sigma_\theta$. This is the case for isotropic compression and triaxial compression (that is, vertical loading) of the hollow cylinder specimen in which $\epsilon_r = \epsilon_\theta$. However, Eqs 21 and 22 show that if torsion shear stress, $\tau_{\theta z} = \tau_{z\theta}$ (see Fig. A), are applied to the hollow cylinder specimen in which $\sigma_r = \sigma_\theta$, then $\partial I_3/\partial\sigma_r$ and $\partial I_3/\partial\sigma_\theta$ would be different, because $\tau_{\theta z} = \tau_{z\theta} \neq 0$ and $\tau_{zr} = \tau_{rz} = 0$. Consequently, the values of $\dot{\epsilon}_r^p$ and $\dot{\epsilon}_\theta^p$ would be different for a material whose description requires the presence of the third stress invariant I_3 (or similar) in the plastic potential function.

In general, and from an objective experimental point of view, it would not be reasonable for deny the occurrence of I_3 in the plastic potential function. In particular, experimental evidence for frictional materials has clearly shown that I_3 (or similar) plays a key role in the description of these types of materials. The presence of I_3 in failure criteria, yield criteria, and plastic potential functions accounts for the smooth triangular cross sections in octahedral planes which are so characteristic of frictional materials such as sand, clay, concrete, rock, and ceramic.

In conclusion, it does not appear to be appropriate to set $\epsilon_r = \epsilon_\theta$ a priori for torsion shear tests on soil. These two horizontal normal strains should clearly be different for any frictional material tested in a hollow cylinder specimen under usual torsion shear conditions, especially near failure where the plastic behavior dominates the elastic behavior.

A. S. Saada (*author's closure*)—Professor Lade is quite correct in trying to put an emphasis on the fact that assuming $\epsilon_r = \epsilon_\theta$ results in restrictions being placed on the potential function of classical plasticity theory, when such a theory is used. Indeed the author himself makes such a statement in his presentation. In the elastic stage, Eqs 1 to 8 of the discussion are quite appropriate. However, for the case of a linearly elastic hollow cylinder subjected to the same inner and outer pressure, with or without torsion, $\epsilon_r = \epsilon_\theta$. The cylinder does not have to be thin walled, and no assumption has to be made on whether the variation of ϵ_r across the thickness is negligible or not. This can be directly seen from the expression of u_r which can be found in any elasticity book (see Ref *1* for example). Indeed, ϵ_r is simply a constant! So is ϵ_θ. Let us not forget however that $\tau_{\theta z}$ and $\gamma_{\theta z}$ vary linearly across the thickness, and that there is an approximation involved in assuming their uniform distribution. This assumption decreases in importance as one proceeds into the plastic stage.

In the plastic stage, the author would like to write the equations in a way slightly more general than that of the discusser. Using classical plasticity,

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\frac{\partial g}{\partial\sigma_{ij}}$$

If we assume that the potential g is an isotropic function of the state of stress, this function must depend only on the invariants, so that

$$g = g\,(I_1, I_2, I_3)$$

and

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\left(\frac{\partial g}{\partial I_k}\frac{\partial I_k}{\partial\sigma_{ij}}\right)$$

where

$$\frac{\partial I_1}{\partial\sigma_{ij}} = \delta_{ij},\ \frac{\partial I_2}{\partial\sigma_{ij}} = 2\sigma_{ij},\ \frac{\partial I_3}{\partial\sigma_{ij}} = 3\sigma_{it}\,\sigma_{tj}$$

The flow rule is of the form,

$$\dot{\underset{\sim}{\epsilon}}^p = \dot{\lambda}\,(A\,\underline{\delta} + B\,\underset{\sim}{\sigma} + C\underset{\sim}{\sigma}^2) \tag{1}$$

where A, B, and C are constants.

If on the other hand $\partial g/\partial I_3 = 0$, $g = g\,(I_1, I_2)$ and

$$\dot{\underset{\sim}{\epsilon}}^p = \dot{\lambda}\,(A\,\underset{\sim}{\delta} + B\,\underset{\sim}{\sigma}) \tag{2}$$

It is the derivative of the third invariant that brings in the square of the stress tensor $\underset{\sim}{\sigma}^2$. If $\sigma_r = \sigma_\theta = p$, the pressure in cell, Eq 2 gives

$$\dot{\epsilon}_r^p = \dot{\epsilon}_\theta^p = \dot{\lambda}\,(A + Bp) \tag{3}$$

This equality only takes place if $\partial g/\partial I_3 = 0$, that is, if the cross section of the yield function by the π plane is circular (similar to Von Mises or Prager-Drucker). On the other hand, Eq 1 gives

$$\dot{\epsilon}_r^p = \dot{\lambda}\,(A + Bp + Cp^2) \tag{4}$$

$$\dot{\epsilon}_\theta^p = \dot{\lambda}\,[A + Bp + C\,(p^2 + \tau_{\theta z}^2)] \tag{5}$$

As one can see $\dot{\epsilon}_r^p \neq \dot{\epsilon}_\theta^p$ when the torsional stresses are applied to the sample, and g is also a function of the third invariant.

There also is another equation that does not depend on the law of behavior of the material and that is the compatibility equation:

$$\frac{\partial \epsilon_\theta}{\partial r} + \frac{\epsilon_\theta - \epsilon_r}{r} = 0 \tag{6}$$

If one assumes that ϵ_θ does not vary with the radius then $\epsilon_r = \epsilon_\theta$. Is that assumption better or worse than assuming that $\tau_{\theta z}$ is uniform and does not vary with the radius? After all, in both cases we chose some uniform value which is obtained, at least in theory, by an averaging procedure and assume that it is constant across the thickness.

Experiments seem to show that at "failure" the trace of the failure surface on the π plane is not exactly circular and is closer to the shape of the coulomb criterion, but with curved segments. In fitting the data by plotting increments of strain, how accurate can one be in distinguishing between a perfectly circular segment of the yield surface or just a curved one? To simplify their calculation some modellers keep two invariants only; are they farther from reality than those who use the three invariants? Moreover, let us not forget that the assumption that g can be expressed in terms of the invariants implies an isotropic material which, more often than not, is far from being the case.

Finally, the above discussion is aimed at the situation where classical plasticity is used. There are other theories that do not use plastic potentials and where the assumption that $\epsilon_r = \epsilon_\theta$ does not create particular hardships: elasto-plastic rules are not the only ones in use today.

When it comes to comparing predicted and measured strains, one may accept as a temporary compromise to compare the measured and predicted values of $(\epsilon_r + \epsilon_\theta)$ which can be obtained without difficulty and quite accurately in most tests.

Bibliography

Saada, A. S., *Elasticity, Theory and Applications* Krieger, Malabar, FL, 1983.