## APPENDIX IV

## THE WEIBULL DISTRIBUTION FUNCTION FOR FATIGUE LIFE*

On the assumption that fatigue failures are initiated at the "weakest link," the fatigue lives of a group of specimens tested under a given set of conditions may be represented by one of a family of frequency distribution functions:

$$f(N) = \frac{b}{N_a - N_o}\left[\frac{(N - N_o)}{(N_a - N_o)}\right]^{b-1} \exp\left\{-\left[\frac{N - N_o}{N_a - N_o}\right]^{b}\right\}$$

where:

$N$ = specimen life,
$N_o \geqq 0$ = minimum life parameter,
$N_a$ = characteristic life parameter occurring at the 63.2 per cent failure point for the population [$63.2 = 100(e - 1/e)$, $e = 2.718$], and
$b > 0$ = Weibull shape (or "slope") parameter.

This function is a simple exponential distribution function when $b = 1$; the Rayleigh distribution function when $b = 2$; and a good approximation of the Normal distribution function when $b = 3.57$, that is, when the mean and the median values are equal.

The curve representing this function (Fig. 12) is usually skewed to the right, going on to infinity, and, for $b > 1$, reaches zero frequency (touches the life axis) to the left of the mode, which is the life value where the highest number of failures occur.

The distribution is said to have a nonzero minimum life if the curve touches the life axis at a value of life greater than 0. In other words, any specimen from the population represented by such a distribution will have zero probability of

* This description of the Weibull distribution function, as an addition to *ASTM STP 91-A*, was originally prepared by a Task Group in Subcommittee VI on Statistical Aspects of Fatigue of ASTM Committee E-9 on Fatigue, composed of: C. A. Moyer, chairman, Physical Laboratories, Timken Roller Bearing Co.; J. J. Bush, General Motors Research Laboratories; and B. T. Ruley, New Departure Div., General Motors Corp. It has been revised, prior to publication in its present form, by another Task Group in Subcommittee VI, composed of: John K. H. Kao, chairman, New York University; Robert A. Heller, Columbia University; B. T. Ruley, New Departures Div., General Motors Corp.; J. M. Holt, Applied Research Laboratories, U. S. Steel Corp.; M. P. Semenek, International Harvester Co.; and G. R. Gohn and Miss M. N. Torrey, Bell Telephone Laboratories, Inc.

There has been a demand from the roller bearing industry for the inclusion of an additional section covering the use of the extreme-value distribution originally proposed for the analysis of fatigue data by W. Weibull (**31,32**). Since Fisher and Tippett (**33**) are often credited with first showing that this distribution was one of three limiting types of the extreme-value distribution, it is sometimes referred to as "Fisher-Tippett Type III for smallest values." As pointed out by Freudenthal and Gumbel (**34**), this distribution has some theoretical basis, assuming that fatigue failures are examples of extreme values, that is, they are smallest-strength or weakest-link values. It has also been used by others in the analysis of life test data.

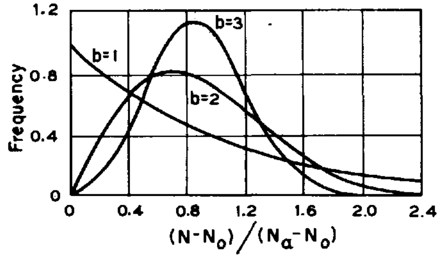



FIG. 12.—Typical Weibull Distribution Curves, from Kao (35).

TABLE 35.—ORDINATE LOCATIONS CORRESPONDING TO PER CENT FAILED VALUES.

| $F(N) \times 100$ | $\log \frac{1}{1 - F(N)}$ | $F(N) \times 100$ | $\log \frac{1}{1 - F(N)}$ |
|---|---|---|---|
| 2 | 0.0088 | 52 | 0.3188 |
| 4 | 0.0177 | 54 | 0.3372 |
| 5 | 0.0223 | 55 | 0.3468 |
| 6 | 0.0269 | 56 | 0.3565 |
| 8 | 0.0362 | 58 | 0.3768 |
| 10 | 0.0458 | 60 | 0.3979 |
| 12 | 0.0555 | 62 | 0.4202 |
| 14 | 0.0655 | 63.2 | 0.4341 |
| 15 | 0.0706 | 64 | 0.4437 |
| 16 | 0.0757 | 65 | 0.4559 |
| 18 | 0.0862 | 66 | 0.4685 |
| 20 | 0.0969 | 68 | 0.4949 |
| 22 | 0.1079 | 70 | 0.5229 |
| 24 | 0.1192 | 72 | 0.5528 |
| 25 | 0.1249 | 74 | 0.5850 |
| 26 | 0.1308 | 75 | 0.6021 |
| 28 | 0.1427 | 76 | 0.6198 |
| 30 | 0.1549 | 78 | 0.6576 |
| 32 | 0.1675 | 80 | 0.6990 |
| 34 | 0.1805 | 82 | 0.7447 |
| 35 | 0.1871 | 84 | 0.7959 |
| 36 | 0.1938 | 85 | 0.8239 |
| 38 | 0.2076 | 86 | 0.8539 |
| 40 | 0.2218 | 88 | 0.9208 |
| 42 | 0.2366 | 90 | 1.000 |
| 44 | 0.2518 | 92 | 1.097 |
| 45 | 0.2596 | 94 | 1.222 |
| 46 | 0.2676 | 95 | 1.301 |
| 48 | 0.2840 | 96 | 1.398 |
| 50 | 0.3010 | 98 | 1.699 |

NOTE.—All logs are to the base 10.

failure prior to $N_o$ life. Later it will be shown how to test for $N_o$ values greater than zero, but if it is reasonable to assume $N_o = 0$, the frequency distribution function is simplified.

Since the data are usually obtained in an ordered manner in fatigue testing, it is easy to fit a cumulative distribution function to fatigue life. The cumulative function for the fraction of population failed prior to life $N$ is

$$F(N) = 1 - \exp\left\{-\left[\frac{N - N_o}{N_a - N_o}\right]^b\right\}$$

This function can be transformed into the straight-line relationship

$$\log \log \left[\frac{1}{1 - F(N)}\right] = b \log (N - N_o) - b \log (N_a - N_o) - 0.36222$$

which allows a simple graphical method for fitting the Weibull distribution to the data and the subsequent graphical estimation of the parameters ($b$, $N_o$, and $N_a$) in the formula.

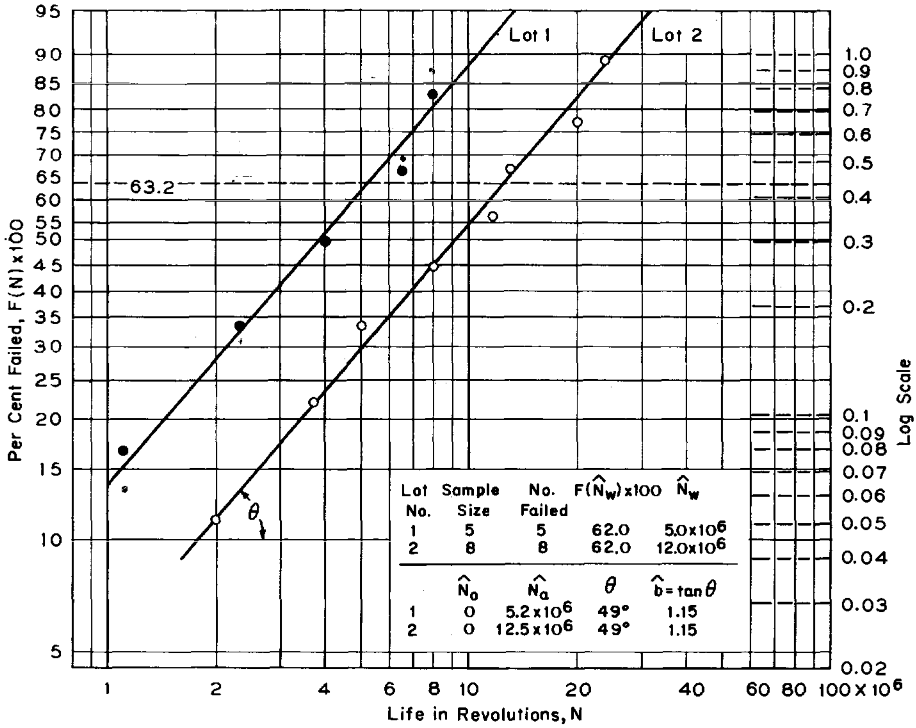



FIG. 13.—Construction of Weibull Probability Paper from Log-Log Paper.

**Construction of Probability Paper**

Although Weibull probability paper can be purchased from a source such as Cornell University, Ithaca, N. Y., Columbia University, New York, N. Y., or Technical and Engineering Aids for Management, 104 Belrose Ave., Lowell, Mass., it can be constructed rather simply from square log-log paper, that is, log-log paper in which the cycles are the same size in both directions. The paper is prepared by the marking off on the vertical logarithmic scale of the probability percentages $F(N)$ corresponding to the values of

$$\log \left[\frac{1}{1 - F(N)}\right]$$

given in Table 35. For example, in Fig. 13, the ordinate of the 90 per cent failure value is 1.000 on the vertical logarithmic scale. Similarly, the ordinate for the 20

TABLE 36.—MEAN-RANK ESTIMATES[a] OF THE PER CENT POPULATION FAILED CORRESPONDING TO FAILURE ORDER IN SAMPLE.

| Order No., $q$ | Sample Size, $n$. | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| No. 1 | 50.00 | 33.33 | 25.00 | 20.00 | 16.67 | 14.29 | 12.50 | 11.11 | 10.00 | 9.09 | 8.33 | 7.69 | 7.14 | 6.67 | 6.25 | 5.88 | 5.56 | 5.26 | 5.00 | 4.76 |
| No. 2 | ... | 66.67 | 50.00 | 40.00 | 33.33 | 28.57 | 25.00 | 22.22 | 20.00 | 18.18 | 16.67 | 15.38 | 14.29 | 13.33 | 12.50 | 11.76 | 11.11 | 10.53 | 10.00 | 9.52 |
| No. 3 | ... | ... | 75.00 | 60.00 | 50.00 | 42.86 | 37.50 | 33.33 | 30.00 | 27.27 | 25.00 | 23.08 | 21.43 | 20.00 | 18.75 | 17.65 | 16.67 | 15.79 | 15.00 | 14.29 |
| No. 4 | ... | ... | ... | 80.00 | 66.67 | 57.14 | 50.00 | 44.44 | 40.00 | 36.36 | 33.33 | 30.77 | 28.57 | 26.67 | 25.00 | 23.53 | 22.22 | 21.05 | 20.00 | 19.05 |
| No. 5 | ... | ... | ... | ... | 83.33 | 71.43 | 62.50 | 55.56 | 50.00 | 45.45 | 41.67 | 38.46 | 35.71 | 33.33 | 31.25 | 29.41 | 27.78 | 26.32 | 25.00 | 23.81 |
| No. 6 | ... | ... | ... | ... | ... | 85.72 | 75.00 | 66.67 | 60.00 | 54.55 | 50.00 | 46.15 | 42.86 | 40.00 | 37.50 | 35.29 | 33.33 | 31.58 | 30.00 | 28.57 |
| No. 7 | ... | ... | ... | ... | ... | ... | 87.50 | 77.78 | 70.00 | 63.64 | 58.33 | 53.85 | 50.00 | 46.67 | 43.75 | 41.18 | 38.89 | 36.84 | 35.00 | 33.33 |
| No. 8 | ... | ... | ... | ... | ... | ... | ... | 88.89 | 80.00 | 72.73 | 66.67 | 61.54 | 57.14 | 53.33 | 50.00 | 47.06 | 44.44 | 42.11 | 40.00 | 38.10 |
| No. 9 | ... | ... | ... | ... | ... | ... | ... | ... | 90.00 | 81.82 | 75.00 | 69.23 | 64.29 | 60.00 | 56.25 | 52.94 | 50.00 | 47.37 | 45.00 | 42.86 |
| No. 10 | ... | ... | ... | ... | ... | ... | ... | ... | ... | 90.91 | 83.33 | 76.92 | 71.43 | 66.67 | 62.50 | 58.82 | 55.56 | 52.63 | 50.00 | 47.62 |
| No. 11 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 91.67 | 84.62 | 78.57 | 73.33 | 68.75 | 64.71 | 61.11 | 57.89 | 55.00 | 52.38 |
| No. 12 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 92.31 | 85.71 | 80.00 | 75.00 | 70.59 | 66.67 | 63.16 | 60.00 | 57.14 |
| No. 13 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 92.86 | 86.67 | 81.25 | 76.47 | 72.22 | 68.42 | 65.00 | 61.90 |
| No. 14 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 93.33 | 87.50 | 82.35 | 77.78 | 73.68 | 70.00 | 66.67 |
| No. 15 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 93.75 | 88.24 | 83.33 | 78.95 | 75.00 | 71.43 |
| No. 16 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 94.12 | 88.89 | 84.21 | 80.00 | 76.19 |
| No. 17 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 94.44 | 89.47 | 85.00 | 80.95 |
| No. 18 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 94.74 | 90.00 | 85.71 |
| No. 19 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 95.00 | 90.48 |
| No. 20 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 95.24 |

[a] Mean-rank estimates $= 100 \dfrac{q}{(n + 1)}$.

per cent failure line is 0.0969 on the logarithmic scale. On such paper, the tangent of the angle $\theta$ is an estimate of the Weibull "slope," $b$, for the population line. The angle $\theta$ may be measured with a protractor, or the slope of the line may be computed.

**Plotting Positions on Probability Paper:**

The fatigue data for any one sample are first ordered from shortest to longest life, each specimen being given an order number, $q$, from 1 through $n$. The horizontal plotting position is its individual life value. All runouts are assumed to have longer lives than the last ordered specimen that failed, but such data are treated separately below under "Estimates of the Distribution Function Parameters."

The vertical plotting position of the per cent failed (Fig. 13) is the estimate of the per cent of the population failed, $F(N)$, based upon the specimen order number. Mean-rank estimates of the percentages of the population failed at successive order numbers are given in Table 36 for sample sizes ranging from 1 through 20. Mean rank, $q/(n + 1)$, is an unbiased estimate of $F(N)$; such estimates are recommended by Gumbel (36) and Weibull (37). Blom (38) suggests modified mean-rank estimates. For the data given in Table 37 for the sample taken from lot 1, the abscissa for the first specimen is plotted at its life value of $N = 1.1 \times 10^6$ revolutions and the ordinate at $F(N) \times 100 = 16.67$, the plotting position for the first of a sample of five based upon mean ranks given in Table 36.

TABLE 37.—TYPICAL FATIGUE TEST DATA.

| Order, $q$ | Number of Revolutions to Failure | |
|---|---|---|
| | Lot 1 | Lot 2 |
| No. 1... | $1.1 \times 10^6$ | $2.0 \times 10^6$ |
| No. 2... | 2.3 | 3.7 |
| No. 3... | 4.0 | 5.0 |
| No. 4... | 6.5 | 8.0 |
| No. 5... | 8.6 | 11.5 |
| No. 6... | ... | 13.0 |
| No. 7... | ... | 20.0 |
| No. 8... | ... | 23.5 |

TABLE 38.—TYPICAL FATIGUE TEST DATA, WITHOUT RUNOUTS.

Plot of $N$ Versus $F(N)$ Nonlinear

| Order, $q$ | Specimen | Number of Revolutions to Failure |
|---|---|---|
| No. 1... | No. 4 | $4.0 \times 10^5$ |
| No. 2... | No. 2 | 5.0 |
| No. 3... | No. 5 | 6.0 |
| No. 4... | No. 8 | 7.3 |
| No. 5... | No. 1 | 8.0 |
| No. 6... | No. 7 | 9.0 |
| No. 7... | No. 6 | 10.6 |
| No. 8... | No. 3 | 13.0 |

**Estimates of the Distribution Function Parameters:**

1. An estimate of the population cumulative distribution that corresponds to the data plotted in Fig. 13 can be found quickly by drawing a line by eye through the failed points. More refined techniques for calculating this line can be found by referring to Gumbel (36), Lieblein (39), or Kao (35). It is possible to calculate this line by the method of least squares, as illustrated in Section V A4 of this guide. For example:

$$\log \log \left[ \frac{1}{1 - F(N)} \right] = X$$

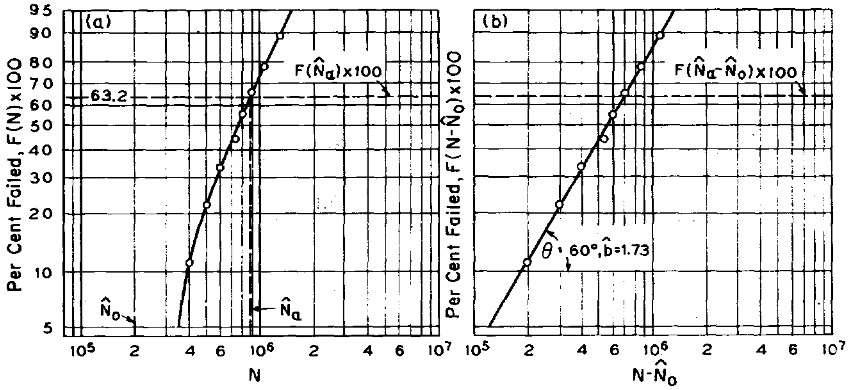
and

$$\log (N - N_o) = Y$$

Comparisons using these methods as against the graphic method show, however, that the latter is usually adequate for small samples.

2. An estimate of the characteristic life, $\hat{N}_a$, is obtained from Fig. 13 by reading off the life value corresponding to the intersection of the fitted line and a horizontal line corresponding to $F(N) \times 100 = 63.2$ per cent.

3. An estimate of the median life is obtained by reading off the life value corresponding to the intersection of the straight line of Fig. 13 and a horizontal line corresponding to $F(N) \times 100 = 50$ per cent.




$$F(N) = 1 - \exp\left[-\left(\frac{N - 2 \times 10^5}{6.9 \times 10^5}\right)\right]^{1.73}$$

$$N_o = 2 \times 10^5$$

$$N_a = 8.9 \times 10^5$$

FIG. 14.—Estimation of Weibull Distribution Function Parameters for Data in Table 38.

4. In Fig. 13, the minimum life, $N_o$, is assumed to equal zero, since the plot of the fatigue data is approximately linear. The plotted data from Table 38 result in a line which curves downward (Fig. 14(*a*)); thus the existence of a finite minimum life value greater than 0 would be suspected. To find an estimate of minimum life, $\hat{N}_o$: (1) note the life value which the curve approaches asymptotically, (2) obtain the quantity $N - \hat{N}_o$ for each point by subtracting the $\hat{N}_o$ value from each individual specimen life, and (3) plot this life difference on Weibull paper versus the same per cent failed values as before. Thus, by trial and error, the best estimate of $\hat{N}_o$ will be found so that the data shown in Fig. 14(*a*) will, when transformed, plot as a straight line, as shown in Fig. 14(*b*).

5. The slope parameter, $b$, is equal to the tangent of the angle $\theta$ shown in Fig. 13. Another estimate of $b$ can be made by computing the tangent of $\theta$ from the logarithms of the ordinates and abscissas of two widely separate points, $N_1$ and $N_2$, on the fitted line. Thus

$$\text{estimate of } b = \frac{\log\log\left[\dfrac{1}{1 - F(N_1)}\right] - \log\log\left[\dfrac{1}{1 - F(N_2)}\right]}{\log(N_1 - N_o) - \log(N_2 - N_o)}$$

6. The skewness of the Weibull distribution varies with the shape parameter, $b$; and the Weibull mean, in general, may occur at various per-cent-failed values;

TABLE 39.—TYPICAL FATIGUE TEST DATA, WITH RUNOUTS.

| Order, $q$ | Specimen | Number of Revolutions to Failure |
|---|---|---|
| No. 1 | No. 2 | $1.30 \times 10^6$ |
| No. 2 | No. 5 | 1.60 |
| No. 3 | No. 4 | 1.75 |
| No. 4 | No. 1 | 2.10 |
| No. 5 | No. 6 | 2.35 |
| No. 6 | No. 3 | 2.70 |
| No. 7 | No. 7 | runout |
| No. 8 | No. 8 | runout |

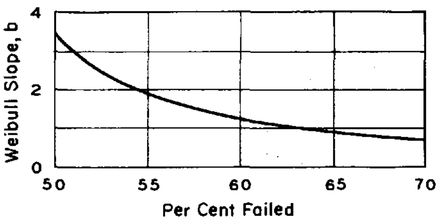
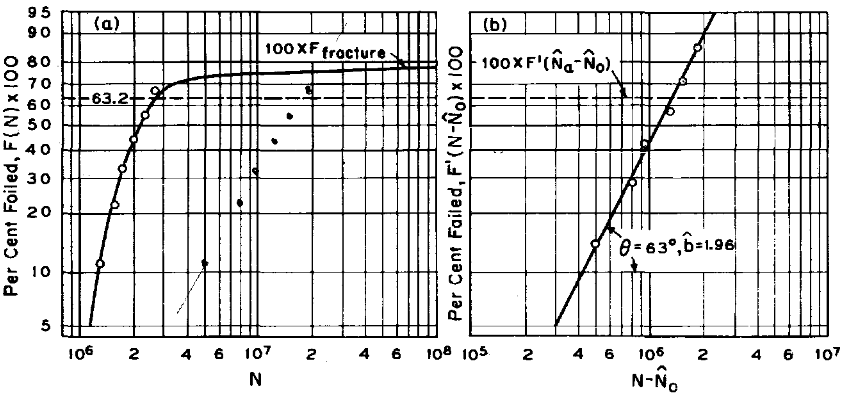



FIG. 15.—Per Cent Failed at Weibull Mean.




$$F_{fracture} = \frac{6 + 1}{8 + 1} = 0.78$$

$$F'(N - \hat{N}_0) = \frac{F(N - \hat{N}_0)}{F_{fracture}}$$

$$F(N) = 0.78 \left\{ 1 - \exp\left[ -\left( \frac{N - 8 \times 10^5}{1.35 \times 10^6} \right)^{1.96} \right] \right\}$$

$$\hat{N}_{63.2} = 2.65 \times 10^6$$

$$\hat{N}_0 = 8 \times 10^5$$

FIG. 16.—Estimation of Weibull Distribution Function Parameters for Data in Table 39.

that is, the mean does not coincide with the median. Using the estimated Weibull slope, $\hat{b}$, it is possible to read from Fig. 15 an estimate of the per cent failed at the Weibull mean and then refer back to the estimated population line on Weibull probability paper, as in Fig. 13, to read off the estimated mean life from the curve. Gumbel (36) and Kao (35) give methods for calculating the Weibull mean[1] when the characteristic life $N_a$ and the slope $b$ are known.

7. For data containing run-out specimens (Table 39), the $n'$ broken specimens (6 in the example, Fig. 16), out of a total of $n$ specimens tested, are plotted on probability paper at the mean-rank plotting positions, corresponding to a sample size $n$ (8 in the example, Fig. 16($a$)). The line drawn through these points will approach a horizontal asymptote, $F_{\text{fracture}}$, which is equal to the ratio of the first plotting positions corresponding to sample sizes $n$ and $n'$, respectively (Fig. 16($a$)).

The parameters of this distribution may be obtained graphically by plotting only the $n'$ broken specimens at mean-rank plotting positions, corresponding to a sample size $n'$ versus $N - \hat{N}_o$, where $\hat{N}_o$ is again the estimate of the vertical asymptote approached by the curve. The slope of the resulting straight line (Fig. 16($b$)), $\tan\theta = \hat{b}$, can be obtained as described in this Section. $\hat{N}_a$, at the probability level of 63.2 per cent, is taken directly from the plotted line. The estimated equation of the probability function for the complete sample of size $n$ will then become

$$F(N) = F_f\left\{1 - \exp\left[-\left(\frac{N - N_o}{N_a - N_o}\right)^b\right]\right\}$$

where $F_f = F_{\text{fracture}}$.

The curve of Fig. 16($a$) may now be replotted by using, as ordinates, $F_{\text{fracture}}$ times the ordinates of the straight line and, as abscissas, $\hat{N}_o$ plus the abscissas of the straight line. Note that $\hat{N}_a$ is, in this case, no longer the estimate of the characteristic life parameter of the complete distribution, $F(N)$. The value of $N$ at the 63.2 per cent probability of failure level may be obtained from the plot in Fig. 16($a$).

[1] Weibull mean:

$$N_w = N_o + (N_a - N_o)\,\Gamma\left(1 + \frac{1}{b}\right)$$

where $\Gamma$ = the gamma function; and for Weibull variance:

$$\sigma^2 = [N_a - N_o]^2\left[\Gamma\left(1 + \frac{2}{b}\right) - \Gamma^2\left(1 + \frac{1}{b}\right)\right]$$