

DISCUSSION

*W. F. Adler*¹—I would like to make a few comments regarding the relation between the one-dimensional momentum balance equations for the propagation of plane waves in laminated materials and the relations used by Dr. Conn and co-workers to evaluate the magnitude of the impact and transmitted stresses for polymeric coatings. First, it is my belief that simple one-dimensional analyses[3,4,6,8] are not general enough to describe the failure mechanisms which actually prevail in coated materials. I also acknowledge that statements appear repeatedly in reports from HYDRONAUTICS that more general approaches are required, but the simplified approach is justified by the fact that the results from the uniaxial plane stress analysis appear to correlate well with the data obtained from erosion tests. After a general development of the governing equation for stress waves propagating in laminated materials, I will discuss certain aspects of the approach used at HYDRONAUTICS in the context in which it was presented.

Consider the one-dimensional shock-wave analysis of laminated materials. The equations for the purely mechanical theory for a water drop striking a coated substrate are given in Fig. 11 for the various plane wave fronts which develop as the pressure pulse propagates into the laminate.

The water drop strikes the coated material with an impact velocity V_0 . A shock wave is transmitted into the drop at a velocity U_w , and a second shock wave is propagated into the coating at a velocity U_c across the water/coating interface. The conditions at the interface between the compressing water drop and coating are that the pressures and particle velocities are continuous; that is, $p_c = p_w$ and $V_c = V_w$. The equation in Fig. 11 corresponds to the balance of mass and momentum as a shock wave is transmitted into the new layer and a second wave is reflected back into the previous layer. Continuity of the pressures and particle velocities also prevails at the coating substrate interface: $p_s = p'_c$ and $V_s = V'_c$. The notation $U[]$ is used to denote that the shock velocity is a function of the particle velocity. This information is available for water. If the relation

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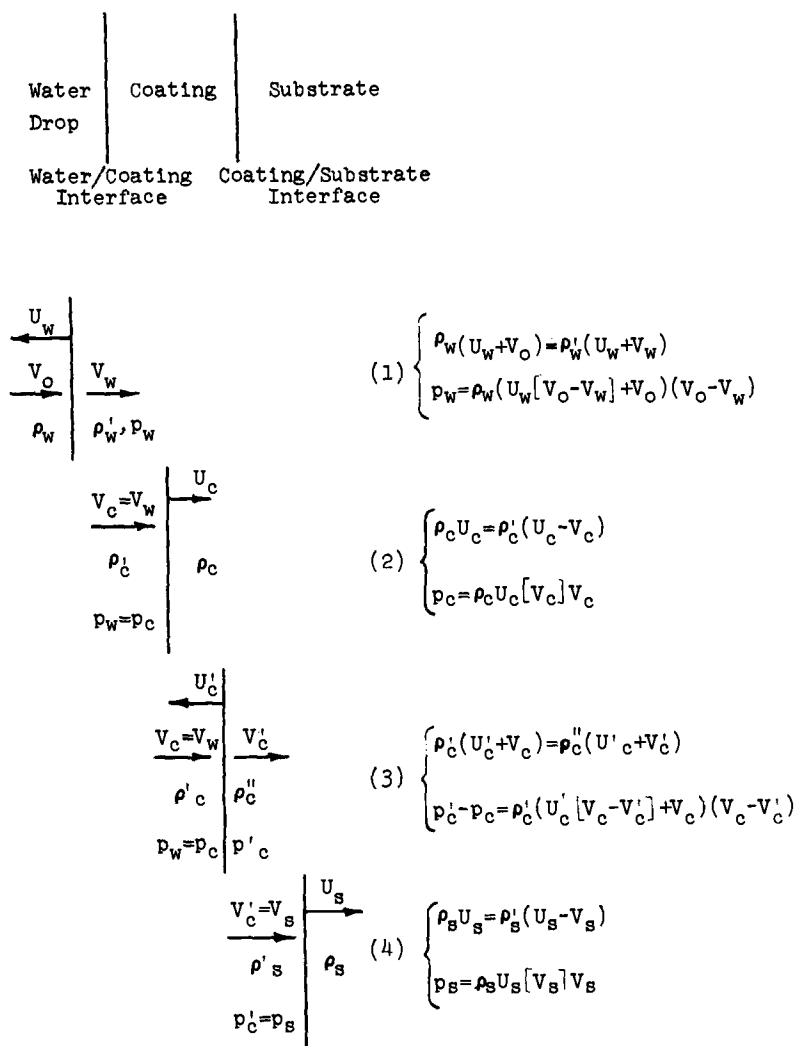


FIG. 11—Mass and momentum balance equations for shock-wave fronts in coated materials.

between the shock velocity and the particle velocity is known for the coating and substrate, the equations provided in Fig. 11 can be solved graphically[8]. Only momentum Eqs 1 and 2 are required to obtain the magnitude of the pressure pulse applied to the coating. The solution of Eqs 3 and 4 can be obtained from the momentum equations in conjunction with the balance of mass from Eq 2 in order to express the density of the

compressed coating, p_r' , in terms of its unstressed density, p_r . Then:

$$p_s = p_c^* + \frac{\rho_c U_c [V_c^*]}{U_c [V_c^*] - V_c^*} (U_c' [V_c^* - V_s] + V_c^*) (V_c^* - V_s) \quad (5)$$

where p_c^* and V_c^* denote the specific values of p_c and V_c obtained from the solution of Eqs 1 and 2.

The approach described in the foregoing was adopted by Morris[8] in estimating the pressure pulse transmitted to the coating and substrate by an impacting water drop. Now consider the form of momentum Eqs 1 to 4 when the droplet impact velocity is small in comparison with the acoustic velocities of the coating and substrate materials. When this condition prevails, the respective particle velocities are also negligible with respect to the acoustic velocities. The shock velocities in Eqs 1 to 4 become the constant dilatational wave speeds for an elastic medium. Momentum Eqs 1 to 4 become

$$p_w = \rho_w C_w (V_0 - V) \quad (6)$$

$$p_c = \rho_c C_c V \quad (7)$$

where $V = V_c = V_w$ and $p_w = p_r$, and

$$p_s - p_c = \rho_s C_s (V_c - V_s) \quad (8)$$

$$p_s = \rho_s C_s V_s \quad (9)$$

Using the condition that $p_w = p_c$, the unknown velocity V can be eliminated from Eqs 6 and 7 to obtain

$$p_c = \frac{\rho_w C_w V_0}{1 + \frac{\rho_w C_w}{\rho_c C_c}} \quad (10)$$

Similarly, eliminating V_s from Eqs 8 and 9 yields

$$p_s = \frac{2p_c}{1 + \frac{\rho_c C_c}{\rho_s C_s}} \quad (11)$$

The relations given in Eqs 10 and 11 are precisely the expressions used by Dr. Conn and his co-workers[3,4,6] in evaluating the impact stress, σ_I in their notation, and the transmitted stress, σ_T , respectively. The derivation of Eqs 10 and 11 from the momentum equations for uniaxial shock-

wave propagation in a laminated system clearly points out the relation between the governing equations for droplet impacts adopted by Morris[8] and those used in the experimental program at HYDRONAUTICS.[3,4,6]

The assumption that the impact velocities are small in comparison with the speed of propagation of a dilatational wave in the coating or substrate is an inherent limitation on Eqs 10 and 11. We further note that Eq 11 is the same result obtained from the theory of elasticity for a dilatational wave, or a distortional wave, striking the interface between two different media of finite extent at normal incidence.² The quantity ρC , where C is the propagation velocity of an elastic wave, will be referred to as the characteristic impedance of the medium. For an elastic medium and low droplet impact velocities, the only difference in the uniaxial strain theory[3,4,6] is the propagation velocity for an elastic wave (corresponding to a dilatational wave in an extended medium). The wave velocity is given by

$$C^2 = \frac{(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \frac{E}{\rho} \quad (12)$$

for uniaxial strain, and by

$$C^2 = \frac{E}{\rho} \quad (13)$$

for the uniaxial stress theory, where E and ν are Young's modulus and Poisson's ratio, respectively.

For large-amplitude pressure pulses, Dr. Conn replaces Eqs 10 and 11 by

$$p_c = \frac{Z_w V_0}{1 + \frac{Z_w}{Z_r}} \quad (14)$$

$$p_\kappa = \frac{2p_c}{1 + \frac{Z_r}{Z_\kappa}} \quad (15)$$

where Z denotes the dynamic impedance and the subscript denotes the medium to which it is applicable. This notation will be used only to signify the evaluation of the dynamic impedance based on the dynamic stress-strain curve obtained experimentally at HYDRONAUTICS using the SHPB.

² Knott, C. G., "Reflection and Refraction of Elastic Waves," *Philosophical Magazine*, Vol. 48, 1899, pp. 64-97.

I would now like to make some observations regarding the use of Eqs 14 and 15 in the evaluation of polymeric materials using the data supplied in a number of HYDRONAUTICS reports[3,4,6]. First of all, I agree with Dr. Conn that the dynamic response of a material specimen should be considered in describing its behavior when subjected to an erosive environment, unless adequate justification can be provided for considering the process in simpler terms. However, very little consideration has been given to the constitutive behavior of the materials being investigated. In general, polymeric materials will exhibit nonlinear, viscoelastic behavior, which is referred to as elastic-plastic in the context of the uniaxial stress analysis. The viscoelastic nature of polymers subjected to short-duration pulses will not be described here; instead, the discussion provided will follow along the lines developed at HYDRONAUTICS.

On the basis of uniaxial, elastic-plastic stress wave theory:

$$Z = \left(\rho \frac{d\sigma}{d\epsilon} \right)^{1/2} \quad (16)$$

Dr. Conn replaces the dynamic stress-strain curves for polymeric materials by a trilinear approximation which simplifies the evaluation of Eq 16. In his notation

$$Z_1 = \left(\rho \frac{\sigma_1}{\epsilon_1} \right)^{1/2} \quad \text{for } \sigma \leq \sigma_1 \quad (17)$$

$$Z_2 = \left(\rho \frac{\sigma_2 - \sigma_1}{\epsilon_2 - \epsilon_1} \right)^{1/2} \quad \text{for } \sigma_1 < \sigma \leq \sigma_2 \quad (18)$$

$$Z_3 = \left(\rho \frac{\sigma - \sigma_2}{\epsilon - \epsilon_2} \right)^{1/2} \quad \text{for } \sigma_2 < \sigma \quad (19)$$

Now the propagating wave front is no longer plane but for $\sigma_2 < \sigma$ is composed of three plane fronts propagating at wave velocities corresponding to Eqs 17, 18, and 19 such that according to the HYDRONAUTICS data, $C_1 > C_3 > C_2$. This implies that at the higher stress levels (above σ_2) the disturbance will propagate faster than the plane wave corresponding to the range $\sigma_1 \leq \sigma \leq \sigma_2$. I believe that the nonlinear stress-strain relations should possibly be expressed in terms of equivalent stress-equivalent strain plots instead of the engineering stress-strain curves. The equivalent strain measure would be a better representation of the finite strains which develop in the tests on polymers using the SHPB.

Data for the polymers considered at Hydronautics are given in Table 1. I have added a column which provides the values of the wave velocities

TABLE 1—Acoustic and dynamic impedances for polymeric materials.

Material	Density, ρ , $\frac{\text{lb-s}^2}{\text{in.}^4} \times 10^{-4}$	Elastic Modulus, E , psi	$C = \sqrt{\frac{E}{\rho}}$, $\frac{\text{in.}}{\text{s}} \times 10^4$	C (experimental), $\frac{\text{in.}}{\text{s}} \times 10^4$	ρC , $\frac{\text{lb-s}}{\text{in.}^3}$	Z , $\frac{\text{lb-s}}{\text{in.}^3}$
Acrylic plastic	1.12	4×10^5	5.98	7.65	6.70	8.54
Acrylic PVC plastic	1.22	3×10^5	4.96	9.15	6.05	11.1
Unfilled PPO	1.00	4×10^5	6.32	10.9	6.32	10.9
Epoxy laminate	1.70	3×10^5	13.29	14.1	22.6	23.9
Polyurethane (Hill AFB)	0.93	3×10^3	0.57	1.08	0.53	1.00
Polyurethane (AFML)	1.00	9×10^2	0.30	1.45	0.30	1.45
TFE plastic	2.00	9×10^4	2.12	1.22	4.24	2.44

evaluated from Eq 13 corresponding to the quasistatic value of Young's modulus. Now consider a water drop impacting a deformable surface at 1000 mph (1.76×10^4 in./s). Referring to Table 1, it is readily seen that the impact velocity, V_0 , is quite close to the magnitude of the stress wave velocities calculated in terms of either the quasistatic or dynamic Young's moduli for the polymeric materials listed. This condition violates the fundamental assumption used in obtaining Eqs 14 and 15 from the more general momentum balance Eqs 1 to 4. While I definitely question the applicability of the data obtained using the SHPB for the erosion of polymeric materials, I would like to suggest that the evaluation of the impact and transmitted stress be given more careful consideration.

A second point along these lines is that while Dr. Conn and his co-workers have devoted considerable attention to incorporating a dynamic elastic modulus in the evaluation of the stress generated in laminated material systems, they have neglected the compressibility of the water drop, which at moderate impact velocities can have a significant influence on the magnitude of the stress computed from Eq 14. The characteristic impedance of water at an impact velocity of 1000 mph is 8.86 lb-s/in.³ instead of the 5.39 lb-s/in.³ used in the evaluations of coating materials at HYDRONAUTICS. Figure 12 illustrates the relative difference that can occur in the evaluation of the impact stress using Eq 14 with the trilinear approximation to the dynamic stress-strain curve and a number of alternative solutions to momentum Eqs 1 and 2 that can be found in the literature on liquid particle erosion. The solid curves originating at the origin are the trilinear approximation to the stress-particle velocity relations for selected polymers based on experimental data from HYDRONAUTICS. The dashed lines originating at the origin signify the linear stress-particle velocity relations for the same polymers based on the constant values of the acoustic impedance recorded in Table 1. The lines originating at point *A* are the stress-particle velocity relations for water using different values for the impedance. Line \overline{AB} denotes the stress-particle velocity relation used by Conn based on a constant acoustic impedance of 5.39 lb-s/in.³. The line \overline{AC} is the linear stress-particle velocity relation accounting for the compressibility of water at an impact velocity of 1000 mph; the curve \overline{AC} is the nonlinear form of this relation based on the data of Rice and Walsh.³ The line \overline{AD} includes the correction to the dynamic impedance of water as dictated by the form of momentum Eq 1. The corresponding nonlinear form of stress-particle velocity relation is omitted in this case for clarity in Fig. 12.

³ Rice, M. H. and Walsh, J. M., "Equation of State of Water to 250 Kilobars," *Journal of Chemical Physics*, Vol. 26, 1957, pp. 824-830.

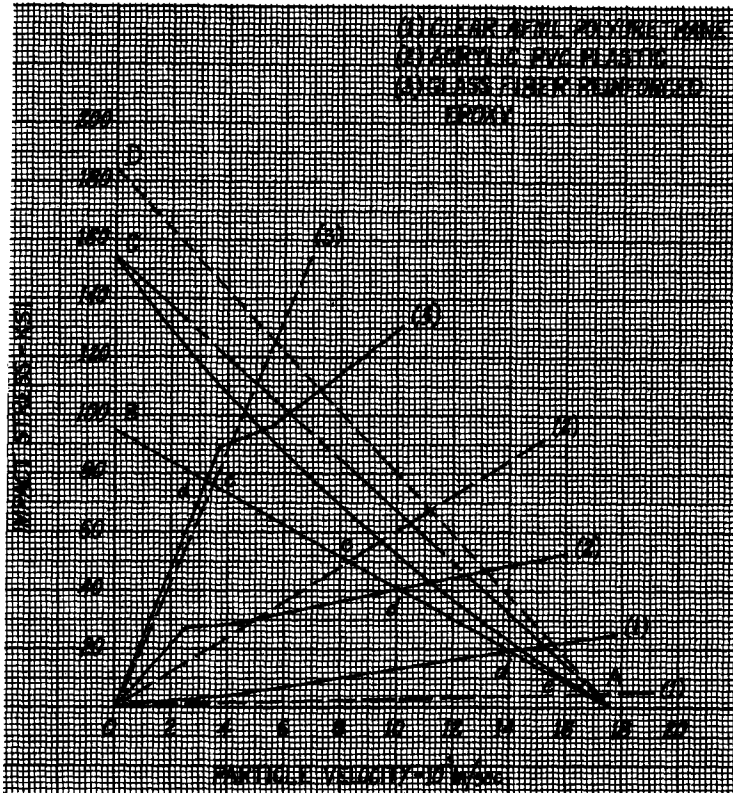


FIG. 12—Impact stress evaluation for a water drop impacting polymeric materials at 1000 mph.

The intersection points in Fig. 12 denoted by d are the evaluation of the impact stress as compared by Conn using Eq 14. The points denoted by e are the straightforward solution of Eq 10. The other points of intersection correspond to various assumptions which can be introduced in the form of the stress-particle velocity relations for water.

The intersection of the nonlinear relation corresponding to line \overline{AD} in Fig. 12, and the nonlinear stress-strain relations for the polymers at strain rates corresponding to the duration of the peak impact pressure for a water drop, would be about the best one-dimensional approximation for the evaluation of the impact stress that could be obtained from Eqs 1 and 2. However, due to the extremely short duration of the peak impact pressure (on the order of $1 \mu s$), the applicability of Eqs 1 and 2 would certainly be questionable since the long wavelength approximation would not be valid under these conditions. The graphical solutions of Eqs 1 and 2 provided in Fig. 12 indicate that the magnitude of the impact stress com-

puted from the dynamic stress-strain curves for polymeric materials can exhibit positive and negative variations from the popular de Haller equation, Eq 10. Figure 12 is also used to demonstrate the large differences which can occur in the results upon introducing the correct form of the momentum balance equations and accounting for the compressibility of water at high impact velocities. The graphical solutions provided for Eqs 1 and 2 can also be carried out for Eqs 3 and 4.

In summary, I feel the following points should be considered in future research employing the uniaxial stress approach:

1. The expressions for evaluating the impact stress and transmitted stress for polymeric materials should be generalized on the basis of the momentum balance equations when the drop impact velocity approaches the wave propagation velocity in the material.

2. Since finite strains are involved in the tests on polymeric materials using the SHPB, the equivalent strain or some other finite strain measure should be used.

3. At droplet impact velocities above 500 mph a more accurate evaluation of the impact stress and transmitted stress can be made within the scope of the one-dimensional analysis if the nonlinear relation between the wave speed and particle velocity for water is taken into account.

A. F. Conn and S. L. Rudy (authors' closure)—The authors are quite flattered and appreciative of the time and effort spent by Dr. Adler in his detailed analysis and commentaries on the experimental (and rather limited analytical) research on rain erosion which has been conducted at HYDRONAUTICS over the past five years. As the main emphasis of our work has been to provide data on the dynamic behavior of the various elastomeric and composite materials encountered in rain erosion situations to our fellow researchers and engineers, it is nice to see that at least one person has indeed read, understood, and begun to use our results.

We want to emphasize that the concern of the sponsor of our studies, the Naval Air Systems Command, is for relatively low-speed rain encounters, that is, up to 500 mph. Thus, we felt that the many other approximations justified using the equations as indicated, and the ambient value for the impedance of water. We fully agree with Dr. Adler in the use of shock wave concepts for the supersonic impacts which torment the aircraft and missile systems traveling at such velocities.

Again, we want to congratulate and thank Dr. Bill Adler for using our research results as the basis for carrying forward by one more step mankind's understanding of this area of rain erosion, and we envy him the time he has available to continue to dig more deeply into this fascinating problem.